An Improved Internal Rate of Return

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Summary

We present here a new way of calculating the internal rate of return of a collection of incomes and expenses using continuous compounding (called IRRCC here). This method has several important advantages over the traditional IRR method. Most significantly there are no restrictions on the timing of the events. That is, after an initial expense, incomes and more expenses can occur in any order and at any time. This is in keeping with the way actual investments work. There is no sense of annual or quarterly compounding, since the compounding is not discrete, but continuous.

It is actually quite simple to state how the method works, because it consists of finding the root of the sum of the values of all of the expenses and incomes corrected to the same time, just like the traditional IRR, but in this case the corrected values are determined by continuous compounding. The root of the sum of the corrected values function cannot in general be found analytically, but, partially owing to the smooth behavior of the exponential function, it readily yields to numerical methods.

Description

Continuous compounding is carried out via the exponential function:

$$FV = PV \times exp(i \times t) \qquad\qquad \text{Eq. 1}$$

Where *FV* is future value, *PV* is present value, *exp* is the exponential function, *i* is interest, typically yearly, and *t* is time as a decimal, again typically in years, measured from the time when *PV* occurred.

This equation can be inverted to convert from future value to present value:

$$PV = FV \times exp(-i \times t) \qquad\qquad \text{Eq. 2}$$

and a collection of *n* future values can all be converted to the same time and summed:

$$f(i) = \sum_{k=1}^{n} A_k \times exp(-i \times t_k) = A_1 + \sum_{k=2}^{n} A_k \times exp(-i \times t_k) \quad \text{Eq. 3}$$

Here we have written the conversion as a function f of interest i, now a variable, and the A_ks are a series of n (signed) amounts, expenses and incomes, occurring at times, t_k, in the future. If we measure the times, t_k, from t_1, then A_1 needs no compounding correction, or, equivalently, $t_1 = 0$.

IRRCC is now the root of f, so that $f(IRRCC) = 0$. In other words, IRRCC is that interest that when substituted for i in f converts all the amounts in our collection to values that sum to zero.

As was mentioned earlier, the exponential function is very smooth and well-behaved, and so is a sum of such functions. For this reason Eq. 3 can be solved by standard numerical methods. I used a variation of Newton's method and convergence is quick.

I have programmed the solution to the IRRCC equation (Eq. 3 above) in Python and JavaScript, where the function is available in a spreadsheet. The user supplies an array of (signed) amounts and the date when each expense or income occurred. The sign choice for expenses is arbitrary (plus or minus), as long as incomes have the opposite sign and consistency is maintained. Curiously, but perhaps not surprising, if you think about it, the amounts and matching times can be presented in any order; summation is commutative.

From the amounts and times, the programs return the IRRCC.

Example

Consider an investment with amounts: -$1000.00, $500.00, -$2000.00, -$2000.00, $1500.00, $4000.00 and dates: 2016-03-16, 2017-09-26, 2018-01-15, 2020-04-05, 2019-05-01, 2021-01-01 .

The IRRCC for this investment is 0.1006 and the plot of our $f(i)$ versus i is:
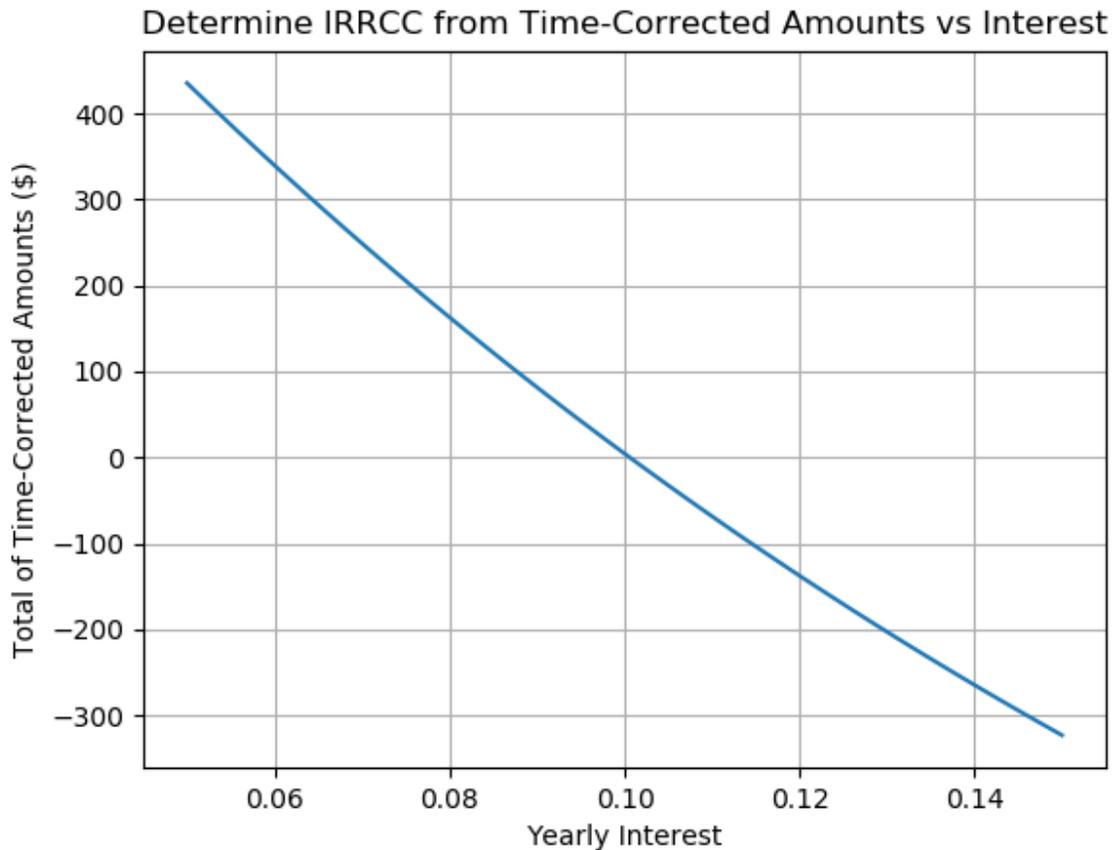
Figure 1: Graph of sum of time-corrected amounts vs interest rate

Note that the root, that is where the function crosses the x axis, agrees with the IRRCC of 0.1006 .

Another Approach

A different way of investigating the IRRCC problem is to break the single summation in Eq. 3 into two sums, one over expenses and the other over incomes, where there are l expenses and m incomes and $l + m = n$. We want to find the interest rate which makes the sum of all of the amounts, positive and negative, equal to zero. If we call the positive amounts *Apos* and the negative amounts *Aneg* Eq. 3 becomes:

$$\sum_{j=1}^{l} Apos_j \times exp(-IRRCC \times t_j) = -\sum_{k=1}^{m} Aneg_k \times exp(-IRRCC \times t_k) \qquad \text{Eq. 4}$$

where we have arbitrarily chosen the *Aneg* to be the expense amounts. In words, when the interest rate is IRRCC, the sum of the time-corrected expenses equals the sum of the time-corrected incomes, but with the signs changed on the expenses. Here is a graph of the same data, divided into two sums, with the sums plotted separately:
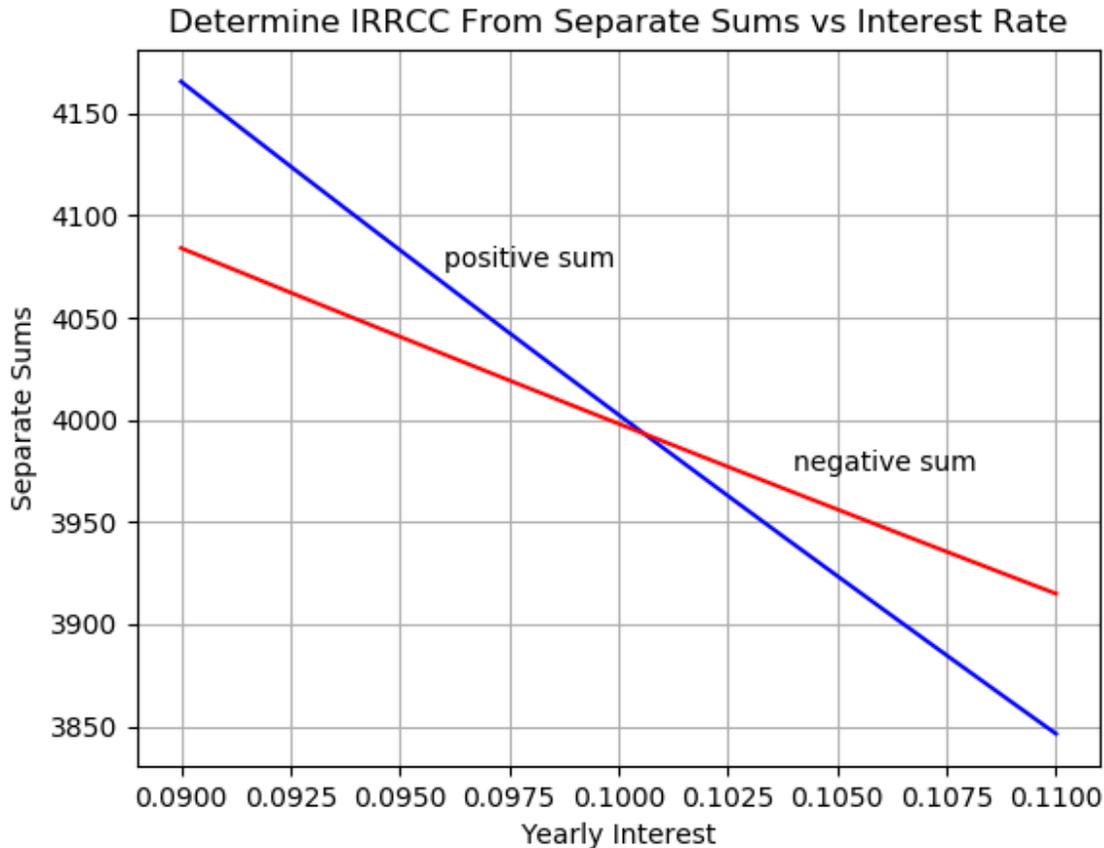
Figure 2: Graph of deposits and withdrawals separately vs interest rate

The lines cross, that is the sums are equal, when the interest rate equals IRRCC, which is 0.1006.

Both sides of Eq.4 are positive, so we can take the logarithm of both sides, bring the right side to the left, and, using the fact that the difference of logarithms is the logarithm of ratios, end up with:

$$log\left(\sum_{j=1}^{l} Apos_j \times exp(-IRRCC \times t_j) / - \sum_{k=1}^{m} Aneg_k \times exp(-IRRCC \times t_k)\right) = 0$$

Eq. 5

Again, in words, when $i = IRRCC$ the sums are equal, so that their ratio is 1, and the logarithm of their ratio is 0. A plot, using natural logs, of Eq. 5 versus i for the same data looks like:
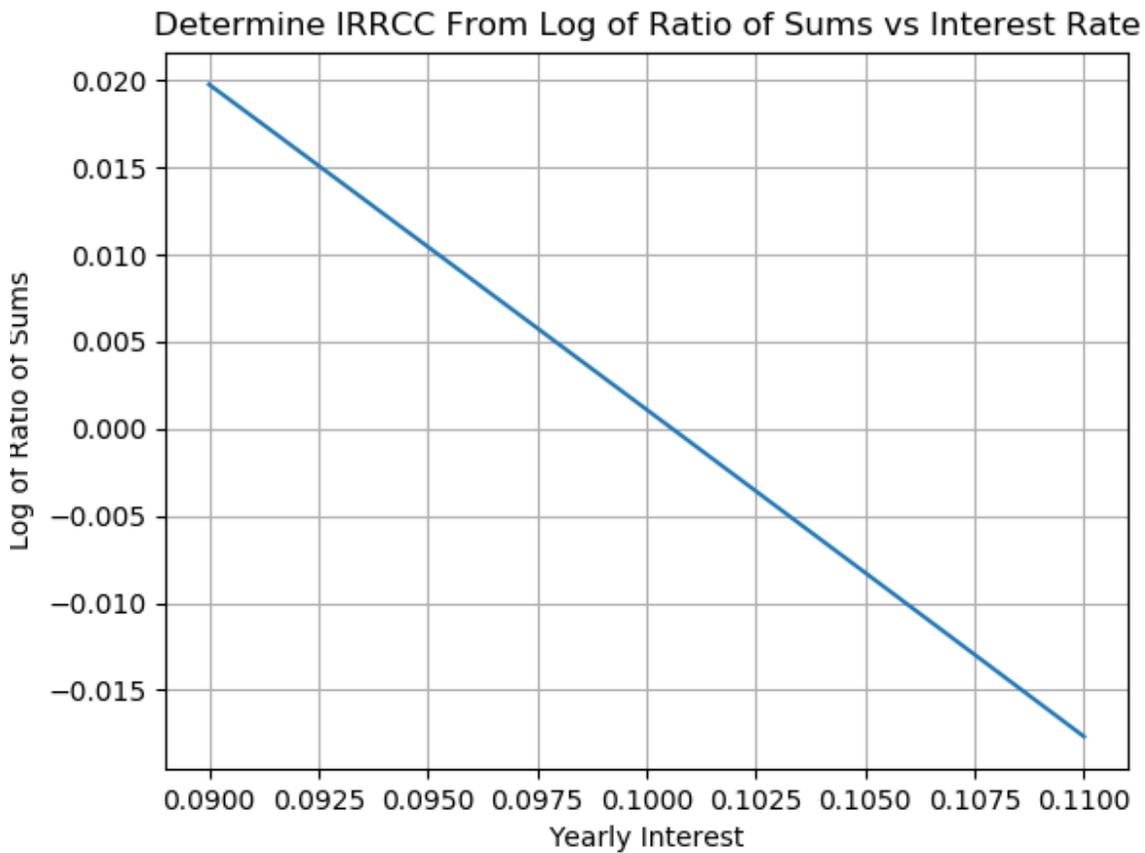
Figure 3: Graph of logarithm of ratio of sums versus interest rate

The logarithm on the vertical axis compresses the curvature seen in Figure 1 and this speeds up Newton's method considerably. Eq. 5 is the equation I solve for IRRCC programmatically. Note the zero crossing at 0.1006 .

A Comparison with IRR

Google Sheets' function for handling irregular Incomes and Expenditures is XIRR, and they give an example in their spreadsheets with this data:

Amounts: -$,8000.00, $1,000.00, $2,500.00, $2,600.00, $3,000.00, $3,800 and Dates: 2010-01-01, 2010-02-01, 2010-04-01, 2010-08-01, 2010-12-01, 2011-02-01.

This investment consisted of one expense of $8,000 followed by five incomes totalling $12,900 in a little over a year (13 months). That's a gain of over 60% in slightly over a year, not taking into account the fact that most of the income occurred more than 11 months after the expense.

For this investment the XIRR is 107.04% and IRRCC is 72.83%, a quite different result.

Even though most of the gains, $6,800, $3,000 11 months after the investment's origination and $3,800 13 months into the investment, XIRR weighs the $1,000 income a month after the original $8,000 expense. the $2,500 income 3 months after the original expense, and the $2,600 income 7 months after the original expense much more heavily than a more gentler appreciating and more realistic IRRCC.

Conclusion

IRRCC is the internal rate of return when continuous compounding is used to convert amounts to different times. It is, like the traditional IRR, that interest rate that when used to convert all the amounts of an investment, expenses and incomes, to the same time, makes the converted expenses and incomes sum to zero. IRRCC differs from IRR only in that amounts are time-shifted using continuous instead of discrete compounding. And of course, expenses and incomes can occur at any time, without being restricted to discrete times.

Continuous compounding is more representative of the way investments behave in the real world, making IRRCC a more realistic measure of the effective interest rate than discrete compounding (IRR), where future amounts jump discontinuously at specific times and are constant at all other times.